1 The Adjoint Representation

Via derivation, the conjugation map $G \to G$, $h \mapsto ghg^{-1}$ induces a representation of a Lie group G and its Lie algebra $\mathfrak{g}$:

Lemma 1.1 (The Adjoint Representation of G) *Let $G \subset GL(n, \mathbb{K})$ be a matrix-group with lie-algebra $\mathfrak{g}$. Then the map*

$$Ad\colon G \to GL(\mathfrak{g}), \quad A \mapsto Ad_A$$

with

$$Ad_A(B) = ABA^{-1}$$

is a representation of G on its Lie-Algebra $\mathfrak{g}$.

Lemma 1.2 (The Adjoint Representation of $\mathfrak{g}$) *Let $\mathfrak{g}$ be a Lie-Algebra. Then the map*

$$ad\colon \mathfrak{g} \to GL(\mathfrak{g}), \quad A \mapsto [A, \cdot]$$

defines a representation of $\mathfrak{g}$ on itself.

The adjoint representations are well behaved with the exponential-map, in the sense that

$$\mathrm{Ad} \circ \exp = \exp \circ \,\mathrm{ad}.$$

2 Compact, Connected Lie Groups

Let G be a compact, connected Lie group. Then the adjoint action restricts to $SO(\dim(G))$:

Theorem 2.1 *Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. Then, after choosing a suitable basis on $\mathfrak{g}$, the adjoint representations of G and $\mathfrak{g}$ become*

$$Ad\colon G \to SO(dim(G)), \qquad ad\colon \mathfrak{g} \to \mathfrak{so}(dim(G))$$

respectively.

The proof uses that every compact Lie group can be embedded in a $U(n)$ for some $n \in \mathbb{N}$ after choosing a suitable basis. Said basis is orthonormal with respect to the Ad-invariant inner product $(\cdot,\cdot)\colon \mathfrak{u}(n) \times \mathfrak{u}(n) \to \mathbb{R}$, $(A, B) = -\mathrm{tr}AB$.

We know more about the adjoint representation:

Lemma 2.1 *Let G be a compact, connected matrix group. Then its adjoint action Ad on $\mathfrak{g}$ has a compact, connected image. Furthermore, its kernel is given by*

$$ker(Ad) = Z(G) = \{h \in G \mid hg = gh \,\forall g \in G\}.$$

3 Root Space Decomposition

Given a compact, connected Lie group G, let $T \subset G$ be a maximal torus with generator g_0. Then the action of T on $\mathfrak{g}$ is determined by $\mathrm{Ad}_{g_0} \in \mathrm{SO}(\dim(G))$. Indeed, we have

$$\mathrm{Ad}_{g_0} \sim \begin{cases} \mathrm{diag}(R(\theta_1), ..., R(\theta_m)), & \text{if } \dim(G) \text{ is even} \\ \mathrm{diag}(R(\theta_1), ..., R(\theta_m), 1), & \text{if } \dim(G) \text{ is odd} \end{cases}$$

for some natural number m and $2d$ rotation matrices R. For each $k = 1, .., m$, define

$$\tilde{\gamma}_k\colon T \to \mathrm{SO}(2), \quad g_0^r \mapsto R(r\theta_k).$$

Then its derivative

$$\gamma_k := D_{id}\tilde{\gamma}_k\colon \mathfrak{t} \to \mathbb{R}$$

defines a **root** of G with respect to T if γ_k is non-trivial. We define the corresponding **root space** $\mathfrak{g}(\pm\gamma_k) \subset \mathfrak{g}$ as the real equivalent of the complex space

$$\bigcap_{t \in \mathfrak{t}} \mathrm{Ker}(\mathrm{Ad}_{\exp(x)} - e^{i\gamma_k(x)}).$$

When γ is non-trivial, we have $\dim \mathfrak{g}(\pm\gamma) = 2m_{\pm\gamma}$ for a natural number $m_{\pm\gamma}$ called the **multiplicity**. If we furthermore set

$$g(0) = \{x \in \mathfrak{g} \mid \forall g \in T,\ Ad_g(x) = x\},$$

then there is the following decomposition of $\mathfrak{g}$:

Theorem 3.1 *Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. Then there is an $\mathbb{R}$-linear direct sum decomposition with respect to the inner product $(\cdot,\cdot)$,*

$$\mathfrak{g} = \bigoplus_{\pm\gamma \text{ root pair}} \mathfrak{g}(\pm\gamma) = \mathfrak{g}(0) \oplus \bigoplus_{\pm\gamma \text{ non-trivial root pair}} \mathfrak{g}(\pm\gamma). \tag{1}$$

The decomposition is called ***root space decomposition*** *of $\mathfrak{g}$.*

Remember the Weyl group $W_G(T) = N_G(T)/T$ acts on $\mathfrak{t}^*$ by $gT \cdot \gamma(x) = \gamma(Ad_{g^{-1}}(x))$. If γ is a root, then so is $gT \cdot \gamma$. This gives us the following theorem:

Theorem 3.2 *Let α, β be roots of G with respect to the maximal torus T. Then the following are true:*

1. *For non-trivial α and $w \in W_g(T)$, the multiplicities of ${}^w\alpha$ and α agree, i.e., $m_{\alpha_w} = m_w$; in fact, these multiplicities are 1.*
2. *For non-trivial α, the only roots which are non-zero multiples of α are $\pm\alpha$.*
3. *$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$ unless $\alpha+\beta$ is a root, and then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = g_{\alpha+\beta}$ if $\alpha+\beta$ is non-trivial, while $\{0\} \neq [g_\alpha, g_{-\alpha}] \subset \mathfrak{g}_0$.*